

*To the memory of A.B. Migdal*

# Beta decay and other processes in strong electromagnetic fields

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## Abstract

We consider effects of the fields of strong electromagnetic waves on various characteristics of quantum processes. After a qualitative discussion of the effects of external fields on the energy spectra and angular distributions of the final-state particles as well as on the total probabilities of the processes (such as decay rates and total cross sections), we present a simple method of calculating the total probabilities of processes with production of non-relativistic charged particles. Using nuclear  $\beta$ -decay as an example, we study the weak and strong field limits, as well as the field-induced  $\beta$ -decay of nuclei stable in the absence of the external fields, both in the tunneling and multi-photon regimes. We also consider the possibility of accelerating forbidden nuclear  $\beta$ -decays by lifting the forbiddenness due to the interaction of the parent or daughter nuclei with the field of a strong electromagnetic wave. It is shown that for currently attainable electromagnetic fields all effects on total  $\beta$ -decay rates are unobservably small.

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# 1 Introduction

Study of quantum processes in intense electromagnetic fields is a very interesting subject. Strong external fields can help us to learn more about the properties of the involved particles and their interactions. Studying processes in strong fields may also have interesting implications for astrophysics and cosmology. Recently, there has been a renewed interest in this topic in connection with development of new powerful laser sources.

In this paper I will discuss effects of strong external electromagnetic fields on various characteristics of quantum processes. In sec. 2 a rather general qualitative analysis of these effects is given, whereas secs. 3 and 4 are dedicated to a specific example – nuclear  $\beta$ -decay in the field of a strong electromagnetic wave.

My interest in this topic was raised in the early 1980s by A.M. Dykhne, who called my attention to a paper published in the Physical Review Letters [1]. It was claimed in that paper that under the influence of electromagnetic fields of existing at that time powerful lasers  $\beta$ -decay of tritium can be significantly accelerated. Simple estimates I made did not confirm this conclusion, but at the same time I could not pinpoint a mistake in the calculation done in [1]. The problem was that the calculation was very complicated and difficult to follow. It was based on the standard at that time procedure of infinite summation of partial probabilities corresponding to absorption from the external wave (or emission into it) of all possible numbers of photons. This motivated me to look for a simpler way of calculation of the total probabilities of quantum processes in the fields of intense electromagnetic waves. My quest was strongly supported by Victor Khodel, a colleague of mine at the Kurchatov Institute, who used to say that “if there is a simple result, there must exist a simple way of obtaining it”.

Eventually, I found a very simple way of calculating the total probabilities of quantum processes with emission of non-relativistic charged particles. The method in particular allowed one to easily study all interesting limiting cases – the weak and strong field limits as well as the case of the field-induced decay of a particle (or a nucleus) that is stable in the absence of the external field, both in the tunneling and multi-photon limits. The results were published in [2, 3].

While I was working on this subject, several papers appeared [4, 5, 6], where the problem of  $\beta$ -decay in strong electromagnetic fields was re-investigated and it was shown that the results of [1] were erroneous. The analysis in [4, 6] was based on the same old summation technique, whereas the approach in Voloshin’s paper [5] was close in spirit (though not identical) to the one I was developing. In secs. 3 and 4 I will discuss the method of [2, 3] using nuclear  $\beta$ -decay as an example, but it can actually be applied to a much wider class of problems.

Gratifyingly, in all considered limiting cases the results of direct calculations of the probability of  $\beta$ -decay in the field of a strong electromagnetic wave [2, 3] agreed perfectly well with my previously made estimates, often even including the numerical coefficients.

I learned a great deal about how to analyze physics problems qualitatively and how to make simple estimates from A.B. Migdal, both from my personal interactions with him and from his splendid book [7]. It is therefore a great pleasure and honour for me to write this article for the special issue of *Yadernaya Fizika* dedicated to the centennial anniversary of A.B. Migdal's birthday.

## 2 Qualitative considerations

Consider quantum processes such as

- ◇  $A + B \rightarrow A + B$  (scattering)
- ◇  $A + B \rightarrow C + D + \dots$  (reactions)
- ◇  $A \rightarrow B + C + \dots$  (decay)

How can external electromagnetic fields influence these processes? They can

- Modify the differential characteristics of the process (i.e. the energy spectra and angular distributions of final-state particles).
- Affect the total probabilities of the processes, such as total cross sections and decay rates.
- Finally, new channels of reactions or decays, which were not available in the absence of the external fields, may open up.

Let us discuss qualitatively all these possibilities in turn.

We will now make several assumptions that will be used throughout this paper. First, it turns out that the smaller the characteristic energies of the charged particles, the stronger the effects of external electromagnetic fields on the processes in which they are involved. For this reason we shall consider processes with non-relativistic charged particles. We will assume that the system is subjected to the field of a monochromatic electromagnetic wave of frequency  $\omega$  and electric field strength  $\mathbf{E}$ , and that the field is a low-frequency one:

$$\hbar\omega \ll \varepsilon_0, \tag{1}$$

where  $\varepsilon_0$  is the characteristic energy of the process. The low-frequency limit very often also means that the wavelength of the photons of the external field  $c/\omega$  is large compared to the characteristic length of the process  $l_x$  (such as, e.g., the atomic size in photo-ionization processes or the nuclear radius in nuclear  $\beta$ -decay):  $\omega l_x/c \ll 1$ . The assumption that all the participating charged particles are non-relativistic allows us to concentrate only on effects

of the electric component of the external field, whereas the condition  $\omega l_x/c \ll 1$  implies that we can adopt the dipole approximation, in which the external electric field can be considered as a time-dependent uniform field  $\mathbf{E}(t)$ . Such a field can be conveniently described either in the Coulomb gauge

$$A^\mu(t, \mathbf{x}) = (0, \mathbf{A}(t)), \quad \mathbf{E}(t) = -\frac{1}{c} \frac{\partial \mathbf{A}(t)}{\partial t} \quad (2)$$

with the coordinate-independent vector-potential  $\mathbf{A}(t)$ , or in the scalar gauge

$$A^\mu(t, \mathbf{x}) = (\phi(t, \mathbf{x}), 0), \quad \phi(t, \mathbf{x}) = -\mathbf{E}(t)\mathbf{x}, \quad \mathbf{E}(t) = -\nabla\phi. \quad (3)$$

In different situations different gauges turn out to be most convenient for calculations; we will use the Coulomb gauge in sec. 3 and the scalar gauge in sec. 4.

## 2.1 Differential characteristics

Strong external fields can modify energy spectra and angular distributions of particles produced in scattering, reaction or decay process. This happens due to the absorption by a charged particle of photons of the external field (or stimulated emission of photons into it).

Free on-shell particles cannot absorb or emit photons due to energy-momentum conservation. However, a particle that undergoes a scattering which changes its momentum, or is produced in some process (such as an electron production in  $\beta$ -decay or emission of an electron from an atom due to photo-ionization or electron-impact ionization) can exchange photons with the external field.

Let as a result of some process a non-relativistic particle of charge  $e$  and mass  $m$  be produced, and let its kinetic energy be  $\varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0$  is the energy release in the process, i.e. the maximum kinetic energy available to the particle under consideration. The particle can receive some energy from the external field or give to the field a fraction of its energy. Let us estimate the corresponding energy  $\Delta\varepsilon$ . The exchange of the energy between the particle and the field effectively takes place during a characteristic time of order of the period of oscillations of the external field:  $t_{char} \sim 1/\omega$  (the contribution of an integer number of full field periods  $T = 2\pi/\omega$  averages to zero). Therefore the momentum that the particle can obtain from the field is of order

$$\Delta k = eE_0 t_{char} \sim \frac{eE_0}{\omega}, \quad (4)$$

where  $E_0$  is the amplitude of the electric field strength. For particles with a characteristic energy  $\varepsilon_0$  (i.e. with the characteristic momentum  $k_0 = \sqrt{2m\varepsilon_0}$ ) we have

$$\frac{\Delta k}{k_0} = \frac{eE_0}{\sqrt{2m\varepsilon_0}\omega} \equiv \xi. \quad (5)$$

This parameter characterizes, in particular, the modification of the angular distribution of the produced charged particle. If  $\xi$  is not too large, during the characteristic time  $t_{char}$  the particle moves over the distance  $l \sim v_0 t_{char} = \sqrt{2\varepsilon_0/m} t_{char}$ . The energy obtained by the particle from the external field is just the work of the field on the particle over the distance  $l$ , which gives

$$\frac{\Delta\varepsilon}{\varepsilon_0} \sim \frac{eE_0 l}{\varepsilon_0} = \frac{eE_0}{\varepsilon_0} \left( \sqrt{\frac{2\varepsilon_0}{m}} \frac{1}{\omega} \right) = 2\xi. \quad (6)$$

This result is only valid assuming that  $\xi \lesssim 1$ ; for  $\xi \gg 1$  one has to take into account that the velocity of the particle increases with time and is no longer equal to its original velocity  $v_0$ . In this case

$$l \simeq v_0 t_{char} + \frac{eE_0}{m} \frac{t_{char}^2}{2}, \quad (7)$$

where  $eE_0/m$  is the particle's acceleration in the external field. This yields

$$\frac{\Delta\varepsilon}{\varepsilon_0} \sim \frac{eE_0 l}{\varepsilon_0} = \frac{eE_0}{\varepsilon_0} \left( \sqrt{\frac{2\varepsilon_0}{m}} \frac{1}{\omega} + \frac{eE_0}{m} \frac{1}{2\omega^2} \right) = 2\xi + \xi^2. \quad (8)$$

Alternatively, this result can be obtained from  $\Delta\varepsilon \simeq [(k_0 + \Delta k)^2 - k_0^2]/2m$  and eq. (5).

Thus, the modification of the differential characteristics of the process (energy spectra and angular distributions of the produced particles) is governed by the parameter  $\xi$  defined in eq. (5). For  $\xi \gtrsim 1$  the external field sizeably affects these quantities. The total number of photons absorbed from the field (or emitted into the field) for  $\xi \lesssim 1$  can be estimated as

$$N_0 \simeq \frac{\Delta\varepsilon}{\hbar\omega} \simeq \xi \cdot \frac{2\varepsilon_0}{\hbar\omega}. \quad (9)$$

In the considered low-frequency limit (1) it can be very large even for not too strong fields, when  $\xi$  is relatively small.

The above estimates should be taken with some caution, though. The parameter  $\xi$  diverges in the constant-field limit  $\omega \rightarrow 0$ ; does this mean that  $\Delta k$  and  $\Delta\varepsilon$  will diverge as well? In fact, the above estimates of these quantities were made for the field of a plane electromagnetic wave, which in the limit  $\omega \rightarrow 0$  goes into crossed uniform electric and magnetic fields of infinite space-time extension. For non-relativistic particles only the electric field matters, and in an electric field of infinitely large spatial size the quantity  $\Delta\varepsilon$  can indeed formally become arbitrarily large. One should remember, however, that in reality all fields are limited in space and time and, in addition, the distance between the source and detector is finite. This leads to a natural cutoff in the expressions for  $\Delta k/k_0$  and  $\Delta\varepsilon/\varepsilon_0$  in the limit  $\omega \rightarrow \infty$ : the characteristic time  $t_{char} \sim 1/\omega$  should then be replaced by the smaller between the time scale of the field and the particle's time of flight between the source and the detector. The characteristic length  $l$  in eq. (7) has to be modified accordingly.

Note that the parameter  $\xi$  does not contain  $\hbar$ , i.e. is a purely classical quantity; therefore, the distortion of energy spectra and angular distributions of the final-state particles in

external fields is a classical effect even when  $N_0 \sim 1$ . In many interesting cases, however, and in particular for  $\beta$ -decay in strong laser fields, one has  $\hbar\omega \ll \varepsilon_0$ , so that for not too small  $\xi$  one has  $N_0 \gg 1$ . In those cases the exchange of energy between the system and the external field has a multi-photon nature. At the same time, the number of photons  $N_1$  absorbed from the external field (or emitted into it) during the process of the formation of the emitted charged particle may be small. Indeed, the formation process is characterized by the time scale  $t_0 \sim \hbar/\varepsilon_0$ ; therefore the number of photons absorbed or emitted in the course of the particle production process is

$$N_1 \sim \frac{eE_0\sqrt{2\varepsilon_0/m}t_0}{\hbar\omega} = 2\frac{eE_0}{\sqrt{2m\varepsilon_0}\omega} = 2\xi. \quad (10)$$

This means that  $N_1 \sim (\hbar\omega/\varepsilon_0)N_0 \ll N_0$ , i.e. that the number of photons absorbed or emitted in the course of the production process is small compared to the total number of absorbed or emitted photons. The same is true for the energy change: the energy  $\delta\varepsilon$  obtained by the particle from the wave (or given to the wave) during the process of its formation is small compared to the total change  $\Delta\varepsilon$  of the particle's energy:

$$\delta\varepsilon \sim eE_0\sqrt{2\varepsilon_0/m}t_0 \simeq \frac{\hbar\omega}{\varepsilon_0}\Delta\varepsilon \ll \Delta\varepsilon. \quad (11)$$

This means that for  $\hbar\omega/\varepsilon_0 \ll 1$  the distortions of the angular distributions and energy spectra of charged particles in external electromagnetic fields are mostly formed after their production process is already over.

Thus, we have the following picture of how a quantum process in a low-frequency external electromagnetic field occurs. The whole process can be divided into two stages. In the first stage, charged particles are produced in a reaction, decay or scattering process;<sup>1</sup> during the second stage, the produced particles exchange some energy with the external fields, and the observable energy spectra and angular distributions are formed. The two stages are to a large extent independent of each other, though the second stage is, of course, impossible without the first one.

From the above picture it follows that under certain conditions the differential cross sections of the processes should have a form of a product of two factors: the cross section of the process in the absence of the external field and the field-dependent factor describing the exchange of energy between the system and the external field. The second factor is practically independent of the character of the first stage of the process, i.e. is universal.

The condition  $\hbar\omega \ll \varepsilon_0$  which implies that the production or scattering of particles proceeds during times that are much shorter than the oscillation period of the external field, allows one to consider the first stage as a sudden perturbation, or “jarring” of the system in the presence of the external field. Such a concept was developed and analyzed

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<sup>1</sup>Note that a scattering process  $A + B \rightarrow A + B$  with some momentum transfer can be considered as destruction of particles in the initial state and their production in the final state.

in [8], where it was pointed out that for many different processes (such as e.g. stimulated bremsstrahlung, photo-ionization, inverse Compton scattering, etc.) in the field of an external electromagnetic wave the differential cross sections take the form

$$\left(\frac{d\sigma}{d\Omega}\right)_n \simeq \left(\frac{d\sigma}{d\Omega}\right)_0 J_n^2\left(\frac{e\mathbf{E}_0\mathbf{k}}{\hbar m\omega^2}\right), \quad (12)$$

where  $(d\sigma/d\Omega)_0$  is the field-free cross section,  $n$  is the number of photons exchanged with the external field ( $n > 0$  corresponds to absorption of photons from the field and  $n < 0$  to their stimulated emission), and  $J_n(z)$  is the Bessel function. From the well known relation  $\sum_{n=-\infty}^{\infty} J_n^2(z) = 1$  it then follows that the total cross section (i.e. the sum of the partial cross sections corresponding to all possible numbers of emitted or absorbed photons) coincides with the field-free one. In other words, in this approximation the total probability of the process is not modified by the external field,

In fact, the condition  $\hbar\omega \ll \varepsilon_0$  which ensures  $N_1 \ll N_0$  is a necessary but not sufficient condition for the field-independence of the first factor on the right hand side of eq. (12); for this, one would also have to require that the energy exchange between the system and the field in the process of the charged particle production,  $\delta\varepsilon = N_1\hbar\omega$ , be small compared to the characteristic energy of the process  $\varepsilon_0$ . We will discuss this condition in detail in the next subsection.

## 2.2 Field effects on total probabilities of the processes

How can an external electromagnetic field affect the total probabilities (decay rates and cross sections) of quantum processes? There are essentially three possibilities:

- (i) the fundamental interaction responsible for the process gets modified, leading to a modification of the transition operator;
- (ii) the matrix element of the process is altered due to a modification of the wave functions of the involved particles in the external field;
- (ii) the phase-space volume of the process gets changed.

As an example of the first possibility, consider charged-current weak interaction processes (such as e.g. nuclear  $\beta$ -decay) in an external electromagnetic field. Virtual  $W^\pm$  bosons which mediate these processes can interact with the field. However, each act of photon exchange between the  $W$  boson and the field would lead to the appearance of an extra  $W$ -boson propagator in the amplitude of the process, and therefore would strongly suppress it due to the very large mass of the  $W$  boson. A possible exception is the case when the field frequency is extremely high:  $\hbar\omega \gtrsim m_W c^2$ . However, the latter possibility would correspond to a process with the participation of a very hard  $\gamma$ -quantum, i.e. this would be a completely

different process. Yet another possibility is when the external field, though a low frequency one, is very strong. It is easy to see, however, that in order for the field to produce a noticeable effect, the field strength has to be  $E \gtrsim m_W^2 c^3 / e\hbar \simeq 3.2 \times 10^{26}$  V/cm, which is an extremely large value. It is not actually clear if such strong fields may exist in nature.

The possibility (ii) can be realized, e.g., in the case of forbidden nuclear  $\beta$ -decay, where the interaction of nuclei with external electromagnetic fields may change the angular momentum of the wave functions of the initial and/or final nuclear states, thus lifting the forbiddenness of the  $\beta$ -transition. This possibility will be discussed in sec. 4.

Let us now concentrate on the possibility (iii), i.e. on modification of the phase-space volume of the process.

### 2.2.1 Phase space change

Assume that a charged particle produced in a quantum process obtains some energy from the external field. Can this actually increase the phase-space volume of the process and thus influence its total probability? At first sight, this seems to be impossible: Indeed, before the particle is produced, the field cannot affect it; after it has been produced, any change of its energy cannot affect the production probability. You can put the particle into a capacitor or accelerator and accelerate it to a very high speed – this would not modify the production probability because the production process is already over.

However, the above argument is purely classical, as it implies that the production process is instantaneous. Quantum mechanics tells us that in reality the production of a particle with an energy  $\sim \varepsilon_0$  takes a finite time  $t_0 \sim \hbar/\varepsilon_0$ . This is actually a formation time of the particle. The produced charged particle is not pointlike – it is characterized by its de Broglie wavelength:

$$\lambda_D \simeq \frac{\hbar}{k_0} = \frac{\hbar}{\sqrt{2m\varepsilon_0}}. \quad (13)$$

The formation time  $t_0$  can be estimated as the time it takes for the particle's de Broglie wave to emerge from the source (i.e. the time interval over which the particle moves over a distance of order of its de Broglie wavelength):

$$t_0 \sim \lambda_D/v_0 = \frac{\hbar}{\sqrt{2m\varepsilon_0}} \cdot \sqrt{\frac{m}{2\varepsilon_0}} = \frac{\hbar}{2\varepsilon_0}. \quad (14)$$

The energy obtained from the field *during the particle's formation time* can indeed increase the phase space volume of the process and affect its total probability (or cross section). The energy obtained after that has no effect on the total probability – it can only modify the energy spectra and angular distributions of the emitted particles. As discussed in sec. 2.1, the latter effect is purely classical. At the same time, as we have just shown, modification of the total probabilities of processes in external electromagnetic fields is an inherently quantum effect.



Let us now estimate the energy obtained from the field by a particle during its formation time. It is given by the work done by the field on the particle over the characteristic distance  $l_x$  equal to the particle's formation length, which should be of order  $\lambda_D$ . However, after the whole particle's de Broglie wave has emerged from the source, the production process is already over, so for our estimate we take the characteristic length  $l_x$  to be

$$l_x \simeq \frac{\lambda_D}{2}. \quad (15)$$

The energy gain  $\delta\varepsilon_D$  that affects the phase space volume of the process is then

$$\delta\varepsilon_D \simeq eE_0 l_x = \frac{eE_0 \hbar}{2\sqrt{2m\varepsilon_0}}, \quad (16)$$

and the modification of the total probability is governed by the parameter

$$\chi \equiv \frac{\delta\varepsilon_D}{\varepsilon_0} = \frac{eE_0 \hbar}{\sqrt{2m\varepsilon_0} 2\varepsilon_0}. \quad (17)$$

Note that this parameter contains  $\hbar$ , as expected. The modification of total probabilities of quantum processes in the external fields would be substantial provided that  $\chi \gtrsim 1$ .

Let us now give a slightly different argument for this. Assume that a charged particle is produced virtually with the energy  $\varepsilon_0 + \delta\varepsilon$  instead of the energy  $\varepsilon_0$  dictated by energy conservation. Such particle can only exist during a finite time interval  $\tau_0 \sim \hbar/\delta\varepsilon$ ; after that, it must be re-absorbed by its source. However, if during this time interval it receives the missing energy  $\delta\varepsilon$  from the field, i.e.

$$eE \left( \sqrt{\frac{2\varepsilon_0}{m}} \frac{\hbar}{\delta\varepsilon} \right) = \delta\varepsilon, \quad (18)$$

it gets “license to live”, i.e. can become real. To affect the production probability significantly,  $\delta\varepsilon$  must be  $\gtrsim \varepsilon_0$ , which gives

$$\delta\varepsilon/\varepsilon_0 \sim \chi \gtrsim 1, \quad (19)$$

i.e. the same condition as we found before.

Let us stress that the parameter  $\chi$  is independent of  $\omega$  and therefore does not diverge in the constant-field limit  $\omega \rightarrow 0$ , unlike the parameter  $\xi$  defined in (5). This is an important point. In ref. [1] the tritium  $\beta$ -decay process

$${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_e \quad (20)$$

was considered ( $\varepsilon_0 = (M_{{}^3\text{H}} - M_{{}^3\text{He}} - m_e)c^2 = 18.6 \text{ keV}$ ), and it was claimed that in the field of a plane electromagnetic wave the decay rate  $W$  is

$$W \simeq W_0(\varepsilon_0) \cdot [1 + c_0 \xi^2], \quad (21)$$

where  $W_0(\varepsilon_0)$  is the field-free decay rate and  $c_0$  is a number of order unity. Were this result correct, it would mean that one could strongly influence tritium  $\beta$ -decay even by arbitrarily weak fields provided that their frequency is sufficiently small, which is obviously wrong. Moreover, in the limit  $\omega \rightarrow 0$  the result in eq. (21) clearly violates unitarity.

We have pointed out in sec. 2.1 that the parameter  $\xi$  governs the modification of the angular distributions and energy spectra of the particles in the external field, and that the divergence of  $\xi$  in the limit  $\omega \rightarrow 0$  does not pose any problems: the fact that all fields have finite space-time extensions introduces a natural cutoff for very small  $\omega$ . However, one can imagine a situation in which the field occupies a very large space-time region, and in general there is nothing wrong with the fact that, e.g., the energy gain of a charged particle in a constant electric field can be very large if the particle propagates very long distance in the field. For the total probability of the process, this argument would not work: the probability is always limited by unitarity, and therefore field-induced corrections to it cannot contain the field frequency  $\omega$  in the denominator. From our analysis it follows that in relatively weak or moderate fields the tritium  $\beta$ -decay rate should be given by an expression similar to (21), but with  $\xi^2$  replaced by  $\chi^2$  (we will discuss this point in more detail in sec. 2.3). Comparing eqs. (5) and (17) we find

$$\chi = (\hbar\omega/2\varepsilon_0)\xi, \quad (22)$$

i.e. for  $\hbar\omega \ll \varepsilon_0$  one has  $\chi \ll \xi$ .

Recall that the total number of emitted or absorbed photons  $N_0$  and the number  $N_1$  of photons exchanged with the field during the formation of the emitted charged particle are related to the parameter  $\xi$  as  $N_0 \sim \xi(2\varepsilon_0/\hbar\omega) \gg \xi$  and  $N_1 \sim \xi$ . At the same time, the corresponding total energy  $\Delta\varepsilon$  obtained from the field or given to it and the energy  $\delta\varepsilon_D$  exchanged with the field in the course of the particle's production can be expressed as

$$\Delta\varepsilon \simeq N_0 \hbar\omega \simeq 2\xi\varepsilon_0, \quad \delta\varepsilon_D \simeq N_1 \hbar\omega \simeq \chi\varepsilon_0. \quad (23)$$

The above estimates of  $\delta\varepsilon_D$  apply actually to the case of relatively weak external fields, when  $\delta\varepsilon_D \lesssim \varepsilon_0$ , i.e.  $\chi \lesssim 1$ . Let us now consider

### 2.2.2 The strong field limit ( $\chi \gg 1$ )

In deriving eqs. (13)-(16) we were assuming that the de Broglie wavelength of the emitted charged particle is actually fixed by the energy release in the process  $\varepsilon_0$ , i.e. is field-independent. In the case of very strong external fields one has to take into account that the energy gain in the course of the particle formation can be large compared to  $\varepsilon_0$ , i.e. the de Broglie wavelength of the produced particle is in general field dependent. Immediately after the particle production its distance from the source  $l$  is small compared to its de Broglie wavelength. As the particle moves away and its distance from the source increases, its de Broglie wavelength decreases because the particle's momentum  $k$  increases



Figure 1: Schematic representation of emission of the particle's de Broglie wave from the source. Left: weak field case, the de Broglie wavelength is constant. Right: strong field case, the particle gains significant energy and its de Broglie wavelength decreases as the particle moves away from the source.

as  $k \simeq k_0 + eE_0t$  (see fig. 1 where the particle is depicted by a small blob). The total probability of the process is only influenced by the energy that the particle gains over the distance of order of its de Broglie wavelength, i.e. when the increasing  $l$  and decreasing  $\lambda_D$  “meet” each other (more precisely, satisfy the condition  $l = \lambda_D/2$ , see (15)). In the strong field limit we have

$$l \simeq v_0t + \frac{eE_0}{m} \frac{t^2}{2} \simeq \frac{eE_0}{m} \frac{t^2}{2}, \quad (24)$$

$$\lambda_D \simeq \hbar/(k_0 + eE_0t) \simeq \hbar/(eE_0t), \quad (25)$$

and equating  $l$  with  $\lambda_D/2$  we obtain the “meeting time”  $t_1$  and the “meeting coordinate”  $l_1$ :

$$t_1 \simeq \left( \frac{m\hbar}{e^2E_0^2} \right)^{1/3}, \quad l_1 = \frac{\hbar}{2eE_0t_1} = \frac{\hbar}{2eE_0} \left( \frac{e^2E_0^2}{m\hbar} \right)^{1/3}. \quad (26)$$

For the energy obtained by the particle from the field in the course of its production we find

$$\delta\tilde{\varepsilon}_D \simeq eE_0l_1 = \left( \frac{e^2E_0^2\hbar^2}{8m} \right)^{1/3}. \quad (27)$$

The parameter that governs the modification of the total probability of the process in the strong field limit is therefore

$$\frac{\delta\tilde{\varepsilon}_D}{\varepsilon_0} \simeq \frac{eE_0l_1}{\varepsilon_0} = \left( \frac{e^2E_0^2\hbar^2}{8m\varepsilon_0^3} \right)^{1/3} = \chi^{2/3}. \quad (28)$$

Here the tilde over  $\varepsilon_D$  is to distinguish this quantity in the strong field limit ( $\chi \gg 1$ ) from the weak-field value given in eq. (17).

## 2.3 Estimates of total probabilities

We have found that the parameters that describe the effects of external electromagnetic fields on the total probabilities of quantum processes are  $\chi$  in the weak field limit and  $\chi^{2/3}$  in the strong field case. Let us now try to quantify these effects. If the field-free rate of a process is  $W_0(\varepsilon_0)$ , the field-induced increase of the phase-space volume of the process would

imply that the rate of the process in the presence of the field is given by  $W \simeq W_0(\varepsilon_0 + \delta\varepsilon_D)$ . Let us first consider the weak field limit  $\chi \ll 1$ , i.e.  $\delta\varepsilon_D \ll \varepsilon_0$ . In this case we have

$$W \simeq W_0(\varepsilon_0 + \delta\varepsilon_D) = W_0(\varepsilon_0) + W_0'(\varepsilon_0)\delta\varepsilon_D + \frac{1}{2}W_0''(\varepsilon_0)(\delta\varepsilon_D)^2 + \dots \quad (29)$$

The first derivative term, which is linear in the field strength, vanishes upon the averaging over the phase of the field corresponding to the particle production time (or over the angle between  $\mathbf{E}$  and the particle momentum  $\mathbf{k}$ ); the same is also true for all odd-order derivative terms in (29). Assuming that  $W_0$  has a power-law dependence on  $\varepsilon_0$  and taking into account that  $\delta\varepsilon_D \simeq \chi\varepsilon_0$ , we therefore obtain

$$W \simeq W_0(\varepsilon_0) \left[ 1 + \frac{1}{2}W_0''(\varepsilon_0)\chi^2 + \frac{1}{4!}W_0^{(IV)}(\varepsilon_0)\chi^4 + \dots \right]. \quad (30)$$

As an example, consider allowed nuclear  $\beta$ -decay with non-relativistic energy release, such as tritium  $\beta$ -decay. In this case  $W_0(\varepsilon_0) \propto \varepsilon_0^{7/2}$  (see eq. (42) below), and eq. (30) yields

$$W \simeq W_0(\varepsilon_0) \left[ 1 + \frac{35}{8}\chi^2 + \frac{35}{128}\chi^4 + \dots \right]. \quad (31)$$

This expression does not depend on the frequency of the external field, i.e. it actually corresponds to the limit  $\omega \rightarrow 0$ . How should the dependence on  $\omega$  enter into the expression for  $W$ ? The parameter governing this dependence is the ratio of the characteristic time scale of the process  $t_x$  and the field period  $T$ , i.e. it is  $\sim \omega t_x$ . Due to the time reflection invariance of QED, the probability of the process can only depend on the even powers of this parameter. Thus, the coefficients of  $\chi^{2n}$  in the expression for  $W$  should actually be power series in  $(\omega t_x)^2$ . The coefficients in eq. (31) are just the leading (zero order) terms in these expansions. Note that in the considered case  $\chi \ll 1$  we have  $t_x \simeq t_0 \equiv \hbar/2\varepsilon_0$ , so that  $(\omega t_x)^2 = (\hbar\omega/2\varepsilon_0)^2$ .

Consider now the strong field limit  $\chi \gg 1$ , in which  $\delta\tilde{\varepsilon}_D \simeq \chi^{2/3}\varepsilon_0$ . In this case to leading order one can neglect  $\varepsilon_0$  in the expression  $\varepsilon \simeq \varepsilon_0 + \delta\tilde{\varepsilon}_D$  and expand  $W_0(\varepsilon)$  in powers of  $\varepsilon_0/\delta\tilde{\varepsilon}_D = \chi^{-2/3}$ . This gives

$$W \simeq W_0(\varepsilon_0 + \delta\tilde{\varepsilon}_D) \simeq W_0(\varepsilon_0\chi^{2/3}) [1 + \mathcal{O}(\chi^{-2/3})]. \quad (32)$$

For allowed nuclear  $\beta$ -decay with emission of non-relativistic charged leptons, in which  $W_0(\varepsilon_0) \propto \varepsilon_0^{7/2}$ , we find

$$W \simeq W_0(\varepsilon_0) \cdot \chi^{7/3} \cdot [1 + \mathcal{O}(\chi^{-2/3})]. \quad (33)$$

Just like the weak-field expression (31), this result actually corresponds to the limit  $\omega \rightarrow 0$ . The  $\omega$ -dependence of the decay rate  $W$  should be given by the power series in the parameter  $(\omega t_x)^2$ , where the characteristic time  $t_x$  is now given by the “meeting time”  $t_1$  defined in eq. (26). This yields

$$(\omega t_x)^2 = \omega^2 \left( \frac{m\hbar}{e^2 E_0^2} \right)^{2/3} = \chi^{-4/3} \left( \frac{\hbar\omega}{2\varepsilon_0} \right)^2. \quad (34)$$

This means that the  $\omega$  dependence of  $W$  should only enter starting the term of order  $\chi^{-4/3}$  in the expansion (33), whereas the leading term and the term  $\sim \chi^{-2/3}$  in the square brackets should be independent of the frequency of the external field.

Note that the leading term in (33) ( $\sim W_0(\varepsilon_0)\chi^{7/3}$ ) could actually have been guessed. Indeed, since  $W_0(\varepsilon_0) \propto \varepsilon_0^{7/2}$  and  $\chi^{7/3} \propto \varepsilon_0^{-7/2}$ , the leading term in (33) is independent of  $\varepsilon_0$ . This is exactly as it must be in the strong field limit – in this case the field-induced correction to the charged particle energy is large compared to the field-free energy, and the probability of the process should be approximately independent of the latter.

It is interesting to note that the fact that the leading-order term in  $W$  is independent of  $\varepsilon_0$  means that it is also independent of its sign. Therefore, eq. (33) holds true even in the case  $\varepsilon_0 < 0$ , which corresponds to the situation when in the absence of the external field the system is stable (or the reaction does not go because the available energy is below the threshold). For example, a sufficiently strong field can cause  $\beta$ -decay of an otherwise stable nucleus, such as  ${}^3\text{He}$ . The decay rate for this case is also described by eq. (33) provided that by  $W_0$  and  $\chi$  one understands  $W_0(|\varepsilon_0|)$  and  $\chi(|\varepsilon_0|)$ . In other words, a very strong field “pulls” the electron (or positron) out of the nucleus irrespective of whether the nucleus was  $\beta$ -active or stable in the absence of the field.

## 2.4 Weak-field limit for systems stable in the absence of the external field

Consider now in more detail the case of negative energy release ( $\varepsilon_0 < 0$ ), e.g. field-induced  $\beta$ -decay of an otherwise stable nucleus. One can model such a situation by a particle bound in the potential well (fig. 2). In the absence of the external field the particle is in a stationary bound state. The external dipole electric field adds the potential  $U = -eE(t)x$  to the potential well, thus transforming it into a potential with a barrier. The state of the particle then becomes quasi-stationary; the particle can escape from the well either through the tunneling effect or due to a multi-photon “ionization”.

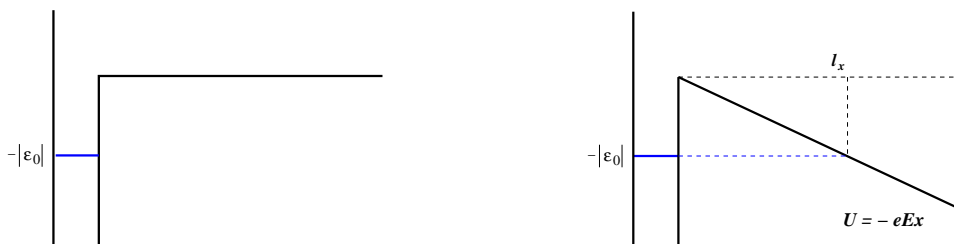


Figure 2: Left: a particle bound in a potential well; right: a particle in a well with a potential barrier.

Let us start with a qualitative analysis, similar to the one performed in the case  $\varepsilon_0 > 0$ .

The  $\beta$ -particle can be emitted (along with the neutrino) virtually, and has to be re-absorbed after the time  $t_0 \sim \hbar/|\varepsilon_0|$ . On the other hand, the time it takes for the electron (or positron) to obtain the missing energy  $|\varepsilon_0|$  from the external field is  $t_2 = \sqrt{2m|\varepsilon_0|}/eE$ . If this time is smaller than or of the order of the “time of virtual existence” of the  $\beta$  particle  $t_0$ , i.e.

$$t_2 = \sqrt{2m|\varepsilon_0|}/eE_0 \lesssim \hbar/|\varepsilon_0|, \quad (35)$$

the induced decay occurs with a large probability ( $W/W_0(|\varepsilon_0|) \gtrsim 1$ ). In the opposite case  $t_2 \gg t_0$  the decay probability is strongly suppressed.

The condition in eq. (35) is equivalent to  $\chi(|\varepsilon_0|) \gtrsim 1$  (see (17)), i.e. corresponds to the strong field limit which was considered in the previous subsection. Let us now concentrate on the weak field limit  $\chi(|\varepsilon_0|) \ll 1$ . In this case the nature of the suppression of the rate of the induced  $\beta$ -decay will depend crucially on the value of  $\omega t_2$ . For  $\omega t_2 \ll 1$  the external field is nearly static, and the field-induced  $\beta$ -decay proceeds as a tunneling effect. From fig. 2 we find that the width of the barrier  $l_2$  satisfies  $eE_0 l_2 = |\varepsilon_0|$ , i.e.

$$l_2 = |\varepsilon_0|/eE_0. \quad (36)$$

The parameter  $t_2$  defined in (35) is then just the tunneling time. The condition  $\chi \ll 1$  means that the WKB approximation can be used, and for the tunneling probability we find

$$W \sim \exp \left[ -\frac{2}{\hbar} \int_0^{l_2} \sqrt{2m(|\varepsilon_0| - eE_0 x)} dx \right] = \exp \left( -\frac{2}{3\chi} \right). \quad (37)$$

Note that this result is non-perturbative in the field strength  $E_0$ . Thus, in the low-frequency limit  $\omega t_2 \ll 1$  the probability of the field-induced  $\beta$ -decay is exponentially suppressed. This result applies quite generally to all processes that do not occur in the absence of the external field. The pre-exponential factor in the expression for  $W$  cannot be found from simple considerations like those presented above. The field frequency dependence of  $W$  should, as usual, be given by the parameter  $(\omega t_x)^2$ , where  $t_x$  now is the tunneling time  $t_2$  defined in eq. (35), which gives

$$(\omega t_x)^2 = \xi^{-2}. \quad (38)$$

Consider now the weak field limit in the case  $\omega t_2 \gg 1$ . The potential barrier then oscillates fast on the time scale of the tunneling time, and the field-induced  $\beta$ -decay proceeds via the absorption of many photons by the  $\beta$ -particle, so that this particle goes over the barrier instead of tunneling through it. The minimum number of the absorbed photons is therefore  $n_1 = \lceil |\varepsilon_0|/\hbar\omega \rceil$ , i.e. is the smallest integer that is  $\geq |\varepsilon_0|/\hbar\omega$ . The probability of the process should then depend on the field strength as

$$W \propto E_0^{2\lceil |\varepsilon_0|/\hbar\omega \rceil}, \quad (39)$$

i.e. it should exhibit a power-law rather than exponential suppression.

### 3 Allowed $\beta$ decay in the field of a strong wave

We shall now go from simple estimates to direct calculations of the total probabilities of quantum processes in a field of a strong electromagnetic wave, taking allowed nuclear  $\beta$ -decay as an example.

Let us first recall the decay rate calculation in the field-free case, According to the Fermi's Golden rule, the decay probability per unit time is given by

$$W_0(\varepsilon_0) = \frac{2\pi}{\hbar} |M_{fi}|^2 \int \frac{d^3k}{(2\pi\hbar)^3} \frac{d^3p}{(2\pi\hbar)^3} \delta(M_i c^2 - M_f c^2 - E_e - E_\nu), \quad (40)$$

where  $\mathbf{p}$  and  $\mathbf{k}$  are the momenta of the emitted neutrino and electron,  $E_e$  and  $E_\nu$  are their full energies, and  $M_i$  and  $M_f$  are the masses of the parent and daughter nuclei. The energy release in the process is  $\varepsilon_0 = (M_i - M_f - m)c^2$ , where  $m$  is the electron mass (we neglect the recoil energy of the final-state nucleus). The matrix element of the process  $M_{fi} = (G_F/\sqrt{2}) \cos \theta_C \mathcal{M}_{fi}$ , where  $G_F$  is the Fermi constant,  $\theta_C$  is the Cabibbo angle, and  $\mathcal{M}_{fi}$  is the nuclear matrix element ( $\mathcal{M}_{fi} = \mathcal{O}(1)$  for allowed  $\beta$ -transitions). The  $\delta$ -function in (40) allows one to remove one of the energy integrations. Let us recall how it appears in eq. (40). The time dependent amplitude of the process is  $\mathcal{A}(t) \propto e^{\frac{i}{\hbar}(E_f - E_i)t}$ . The squared modulus of the integral of the amplitude over the production time therefore yields

$$\left| \int_{-\infty}^{\infty} dt \mathcal{A}(t) \right|^2 \propto \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 e^{\frac{i}{\hbar}(E_i - E_f)(t_1 - t_2)} = T \cdot 2\pi\hbar \delta(E_i - E_f), \quad (41)$$

where we have switched from the integration variables  $t_1$  and  $t_2$  to  $(t_1 + t_2)/2$  and  $\tau \equiv t_1 - t_2$ . The integral over  $(t_1 + t_2)/2$  is trivial and yields the normalization time  $T$  that has to be sent to infinity at the end of the calculations. The decay rate (i.e. the probability per unit time) is obtained by dividing the total probability by  $T$  and therefore is  $T$ -independent. The integral over the difference of  $t_1$  and  $t_2$  gives the  $\delta$ -function  $\delta(E_i - E_f)$ , which helps perform the phase space integration in eq. (40).

In the non-relativistic case ( $\varepsilon_0 \ll mc^2$ ) the direct calculation of (40) yields

$$W_0(\varepsilon_0) = \frac{|M_{fi}|^2}{\hbar^7 c^3} (2m)^{3/2} (4/105\pi^3) \varepsilon_0^{7/2}, \quad (42)$$

i.e.  $W_0 \propto \varepsilon_0^{7/2}$ , as was pointed out above.

How can one calculate the decay rate in the presence of an external electromagnetic wave? Usually, the calculation is done by replacing the standard field-free wave function of the emitted electron with the exact solution of the Dirac equation in the field of a monochromatic electromagnetic wave – the so called Volkov solution [9]. In the non-relativistic case and assuming that the dipole approximation is valid, one can use instead a much simpler wave function [10], which in the Coulomb gauge (2) it can be written as

$$\Psi_{\mathbf{k}}(\mathbf{r}, t) = \exp \left\{ \frac{i}{\hbar} \mathbf{k} \mathbf{r} - \frac{i}{2m\hbar} \int^t [\mathbf{k} - (e/c) \mathbf{A}(t')]^2 dt' \right\}. \quad (43)$$

This is an exact solution of the Schrödinger equation for an electron in a uniform electric field of arbitrary strength and arbitrary time dependence. Since in a non-stationary external field the  $t$ -dependence of the electron wave function is  $\Psi_k(\mathbf{r}, t) \propto e^{-if(t)} \neq e^{-\frac{i}{\hbar}Et}$ , the calculation of the amplitude does not lead to an energy-conserving  $\delta$ -function. This simply reflects the fact that the energy of a system in a time-dependent external field is not conserved. For a periodic electric field  $\mathbf{E}(t)$  with a period  $T = 2\pi/\omega$  the time-dependent factor  $e^{-if(t)}$  in the electron wave function can be expanded in a Fourier series:

$$e^{-if(t)} = \sum_{n=-\infty}^{\infty} a_n e^{-in\omega t}. \quad (44)$$

Substitution of this into the expression for the transition amplitude would lead to a representation of the amplitude as a sum of partial amplitudes corresponding to the absorption from the field or stimulated emission into it of all possible numbers of photons.

Consider a circularly polarized wave with the electric field strength

$$\mathbf{E}(t) = \{E_0 \sin \omega t, -E_0 \cos \omega t, 0\} \quad (45)$$

By making use of the above Fourier expansion technique one can present the decay rate as

$$W = \frac{2\pi}{\hbar} |M_{fi}|^2 \int \sum_{n=-n_0}^{\infty} J_n^2 \left( \frac{eE_0 k_{\perp}}{m\hbar\omega^2} \right) \delta(\varepsilon_0 + n\hbar\omega - \varepsilon_K - \varepsilon - E_{\nu}) \frac{d^3 k}{(2\pi\hbar)^3} \frac{d^3 p}{(2\pi\hbar)^3}, \quad (46)$$

where  $k_{\perp} = \sqrt{k_x^2 + k_y^2}$  is the component of the electron momentum in the plane of the field,  $\varepsilon = k^2/2m$  is the kinetic energy of the electron,  $\varepsilon_K = e^2 E_0^2 / 2m\omega^2$  is the mean kinetic energy of the oscillatory motion of the electron in the field of the wave, and  $n_0 = \lfloor (\varepsilon_0 - \varepsilon_K) / \hbar\omega \rfloor$ , i.e. the integer part of  $(\varepsilon_0 - \varepsilon_K) / \hbar\omega$ . The appearance of the lower limit  $-n_0$  in the sum in (46) is related to the fact that the electron cannot emit into the field more energy than it has. In the limit  $E_0 \rightarrow 0$  only the term with  $n = 0$  in the sum survives and, since  $J_0(0) = 1$ , the field-free probability (40) is recovered.

The expression in eq. (46) was obtained by making use of the standard technique of Fourier expansion of the electron wave function. Now, let us look at this formula more closely. It contains an integral over the electron and neutrino 3-momenta of the sum of squared Bessel functions, whose arguments depend on the electron momentum and on the angle between this momentum and the direction of the external field strength. Moreover, because of the presence of the  $\delta$ -functions, the argument of the Bessel functions  $J_n$  also implicitly depends on the index  $n$  (through the dependence of  $k$  on  $n$ ). The summation extends from a finite value of  $n$  to infinity. Obviously, it would be *very* difficult to calculate the probability of the process using this expression!

It is actually easy to understand why this calculational nightmare arises. The reason for this is a bad calculational approach. Indeed, each term in the sum in (46) gives a partial probability of  $\beta$ -decay with absorption from the field (for  $n > 0$ ) or emission into it



(for  $n < 0$ ) of  $|n|$  photons. The energy conservation for each such processes is ensured by the corresponding  $\delta$ -function in the integrand. In fact, this means that the calculation is done in the energy representation. However, for a system in a non-stationary external field energy is not a good quantum number, therefore using the energy representation does not give us any advantage. Moreover, in order to calculate the total probability of the process, it is actually not necessary to calculate all partial probabilities and then sum them; the partial probabilities actually contain much more information than is needed. Equation (46) therefore uses *excessive* information.

The above arguments actually give us a hint of how an adequate calculation should be done. One should not make use of the Fourier expansion and calculate the partial probabilities. There will be no energy-conserving  $\delta$ -functions in this case, but the integrations over the electron and neutrino momenta can be performed without using  $\delta$ -functions. The idea of a simple calculation presented below is the following:

1. In the calculation of the squared modulus of the integral of the amplitude of the process over the production time go from the variables  $t_1$  and  $t_2$  to  $(t_1 + t_2)/2$  and  $\tau = t_1 - t_2$ , as in the calculation that led to eq. (41).
2. Perform first the integration over  $(t_1 + t_2)/2$ , then over the electron and neutrino 3-momenta, and only at the very end do the integration over the difference of the times  $\tau$ .

Unlike in the field-free case, the integral over  $(t_1 + t_2)/2$  is not trivial now, but it still leads to the proportionality of the total probability of the process to the normalization time  $T$ . For the decay rate (i.e. probability per unit time) we find

$$W = \frac{1}{\hbar^2} |M_{fi}|^2 \int \frac{d^3k}{(2\pi\hbar)^3} \int \frac{d^3p}{(2\pi\hbar)^3} \times \int_{-\infty}^{\infty} J_0\left(2 \frac{e E_0 k_{\perp}}{m\hbar\omega^2} \sin \frac{\omega\tau}{2}\right) \exp\left\{\frac{i}{\hbar}\left(\frac{k^2}{2m} + \frac{e^2 E_0^2}{2m\omega^2} + E_{\nu} - \varepsilon_0\right)\tau\right\} d\tau. \quad (47)$$

Note that at this point it is still easy to go to the standard calculational technique. Indeed, by using the relation [11]

$$J_0(2x \sin y) = \sum_{n=-\infty}^{\infty} J_n^2(x) e^{2iny} \quad (48)$$

one can reduce eq. (47) to (46).

We now perform in (47) the momentum integrations, leaving the integration over  $\tau = t_1 - t_2$  to the end. The integration over the neutrino 3-momentum is trivial,<sup>2</sup> whereas the

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<sup>2</sup>In this calculation the neutrino mass is neglected. We discuss the case  $m_{\nu} \neq 0$  in sec. 3.4.

integral over the electron momentum can be readily done in cylindrical coordinates. The result is [2]

$$W = \frac{\sqrt{\pi i}}{2^5 \pi^4 \hbar^7 c^3} m^{3/2} |M_{fi}|^2 (\hbar \omega)^{7/2} \int_{-\infty}^{\infty} \frac{dx}{(x + i0)^{9/2}} \exp \left\{ -i \left[ \delta \cdot x + \gamma \left( \frac{\sin^2 x}{x} - x \right) \right] \right\}. \quad (49)$$

Here the integration variable is  $x = \omega(t_1 - t_2)/2$ , and we have introduced dimensionless parameters

$$\delta = \frac{2\varepsilon_0}{\hbar \omega}, \quad \gamma = \frac{e^2 E_0^2}{m \hbar \omega^3}. \quad (50)$$

Note that in terms of  $\gamma$  and  $\delta$  the parameters  $\xi$  and  $\chi$  introduced in sec. 2 are expressed as

$$\xi^2 = \frac{\gamma}{\delta}, \quad \chi^2 = \gamma/\delta^3. \quad (51)$$

Equation (49) is the result we were looking for. Instead of the frightening expression (46) we have now obtained a relatively simple single integral. It can be calculated numerically, but all the interesting limiting cases can actually be readily studied analytically.

Let us first note that our approximation (1) implies  $\delta \gg 1$ ; therefore, the integral in (49) gets its main contribution from the region  $|x| \lesssim \delta^{-1} \ll 1$  and from the vicinities of the stationary phase points which lie outside this region. The stationary phase contributions are strongly suppressed fast-oscillating functions of the field strength  $E_0$  which vanish upon averaging over small fluctuations of this quantity; we will discuss their contribution later on. For small  $|x|$  one can expand  $\sin^2 x$  in the exponent in eq. (49) in powers of  $x$ . Keeping the first two terms in this expansion, we find

$$W = \frac{\sqrt{\pi i}}{2^5 \pi^4 \hbar^7 c^3} m^{3/2} |M_{fi}|^2 (\hbar \omega)^{7/2} \int_{-\infty}^{\infty} \frac{dx}{(x + i0)^{9/2}} \exp \left\{ -i \left[ \delta \cdot x - \gamma \frac{x^3}{3} \right] \right\}. \quad (52)$$

The integral here is of Airy type.

### 3.1 Weak-field limit (case $\varepsilon_0 > 0$ )

Consider first the limit

$$\gamma \ll \delta^3 \quad (\chi \ll 1). \quad (53)$$

For brevity, we shall call it ‘the weak-field limit’ even though the field-dependent parameter  $\gamma$  may actually be much greater than unity. For the values of  $x$  satisfying  $|x|\delta \lesssim 1$  one then has  $\gamma|x|^3 \ll 1$ . Therefore in this regime one can expand

$$\exp \left\{ -i \left[ \delta \cdot x - \gamma \frac{x^3}{3} \right] \right\} = e^{-i\delta x} \left( 1 + i\gamma \frac{x^3}{3} + \dots \right). \quad (54)$$

Substituting this into (52), we find

$$W = W_0(\varepsilon_0) \cdot \sum_{m=0}^{\infty} \frac{\Gamma(9/2)}{m! 3^m \Gamma(9/2 - 3m)} \chi^{2m}. \quad (55)$$

The first few terms in this expansion give

$$W = W_0(\varepsilon_0) \left[ 1 + \frac{35}{8}\chi^2 + \frac{35}{128}\chi^4 + \dots \right]. \quad (56)$$

Comparing this with our estimates made in sec. 2 (see eq. (31)), we find a very good agreement. Not only the facts that the decay rate  $W$  depends on the characteristics of the external field through the parameter  $\chi$  and that for small  $\chi$  it is a power series in  $\chi^2$  were predicted correctly by our simple estimates – even the numerical values of the coefficients of the  $\chi^2$  and  $\chi^4$  terms were found correctly. This gives an additional *a posteriori* justification to our choice of the numerical coefficient in the expression for the characteristic length  $l_x$  in eq. (15).

The expression in eq. (56) actually correspond to the stationary field limit  $\omega \rightarrow 0$ . To obtain the dependence of  $W$  on  $\omega$  one has to retain next terms in the expansion of  $\sin^2 x$  in the exponent in eq. (49). For the lowest order terms we obtain

$$W = W_0(\varepsilon_0) \left\{ 1 + \frac{35}{8}\chi^2 \left[ 1 - \frac{1}{30} \left( \frac{\hbar\omega}{2\varepsilon_0} \right)^2 \right] + \mathcal{O}(\chi^4) \right\}, \quad (57)$$

i.e. in the weak field limit the  $\omega$ -dependence enters into the expression for  $W$  as an expansion in powers of  $(\hbar\omega/2\varepsilon_0)^2$ , as it was anticipated in sec. 2.

Let us now turn to the contributions to  $W$  from the stationary phase points. From (52) one finds two stationary phase points:

$$x_{1,2} = \pm(\delta/\gamma)^{1/2} = \pm\xi^{-1}. \quad (58)$$

Their contribution to the decay rate is, to leading order,

$$\delta W = W_0(\varepsilon_0) \frac{105}{16} \chi^4 \cos(2/3\chi). \quad (59)$$

Thus, the stationary phase contributions are suppressed at least as  $\chi^4$  and are fast-oscillating functions of the field strength  $E_0$  with zero average.

### 3.2 Strong field limit ( $\chi \gg 1$ )

In the strong field limit  $\chi \gg 1$  (which corresponds to  $\gamma \gg \delta^3$ ) one can, instead of (54), use the expansion

$$\exp \left\{ -i \left[ \delta \cdot x - \gamma \frac{x^3}{3} \right] \right\} = e^{i\gamma \frac{x^3}{3}} \left( 1 - i\delta x - \frac{\delta^2}{2} + \dots \right). \quad (60)$$

Substituting this into eq. (52) yields

$$W = W_0(\varepsilon_0) \chi^{7/3} \cdot \frac{35}{144} \frac{3^{5/6}}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-3)^m \chi^{-2m/3}}{m!} \sin \left( \frac{2\pi m}{3} + \frac{\pi}{6} \right) \Gamma \left( \frac{m}{3} - \frac{7}{6} \right). \quad (61)$$

This result holds for  $\beta$ -active nuclei ( $\varepsilon_0 > 0$ ). For nuclei that are stable in the absence of the field ( $\varepsilon_0 < 0$ ) the decay rate can be obtained from eq. (61) by replacing  $(-3)^m \rightarrow 3^m$ ,  $W_0(\varepsilon_0) \rightarrow W_0(|\varepsilon_0|)$  and  $\chi(\varepsilon_0) \rightarrow \chi(|\varepsilon_0|)$ .

To obtain the dependence of the decay rate on the field frequency  $\omega$  one has, as usual, to retain the next terms in the expansion of  $\sin^2 x$  in powers of  $x$  in eq. (49). This yields

$$W = W_0(|\varepsilon_0|) \chi^{7/3} \cdot \left\{ \frac{5 \cdot 3^{5/6} \cdot \Gamma(5/6)}{8\sqrt{\pi}} \pm \frac{7 \cdot 3^{1/6} \cdot \Gamma(1/6)}{16\sqrt{\pi}} \chi^{-2/3} + \frac{35\sqrt{3}}{48} \chi^{-4/3} \left[ 1 + \frac{2}{15} \left( \frac{\hbar\omega}{2\varepsilon_0} \right)^2 \right] + \mathcal{O}(\chi^{-2}) \right\}. \quad (62)$$

Here the upper and lower signs correspond to  $\varepsilon_0 > 0$  and  $\varepsilon_0 < 0$ , respectively.

Let us compare this result with the estimates made in sec. 2. It was predicted there that in the strong field limit  $W \simeq W_0(\varepsilon_0) \chi^{7/3}$  times a power series in  $\chi^{-2/3}$  (see (33)). Equation (62) confirms this result, except that the leading term has an extra constant factor  $C_1$ . However, for the numerical value of this factor we have

$$C_1 = \frac{5 \cdot 3^{5/6} \cdot \Gamma(5/6)}{8\sqrt{\pi}} = 0.9943, \quad (63)$$

which is very close to 1. Thus, the predictions based on simple estimates are confirmed once again. The first two terms in the curly brackets in eq. (62) are  $\omega$ -independent, and the dependence of  $W$  on  $\omega$  comes through the positive powers of the quantity  $\chi^{-4/3}(\hbar\omega/2\varepsilon_0)^2$ , again in full agreement with the results of our qualitative analysis in sec. 2. As was discussed there in detail, the strangely-looking dependence of  $W$  on fractional powers of  $\chi$  is a consequence of the fact that in strong fields the electron's de Broglie wavelength is field-dependent.

### 3.3 The case of weak fields and $\varepsilon_0 < 0$

Consider now the case of relatively weak external fields and nuclei stable in the absence of the field ( $\varepsilon_0 < 0$ ). As was discussed in sec. 2, there are two essentially different regimes of the induced  $\beta$ -decay in this case, the tunneling regime and the multi-photon one, depending on whether the field frequency  $\omega$  is small or large compared to the inverse tunneling time  $t_2^{-1} = eE_0/\sqrt{2m|\varepsilon_0|}$ . Both cases can be obtained from the same master equation (49). In the low-frequency limit  $\omega t_2 = \xi^{-1} \ll 1$  in the zeroth order in this parameter one can actually make use of a simpler formula (52). The calculation of the integral in (52) in the steepest descent approximation yields

$$W(\chi) = W_0(|\varepsilon_0|) \frac{105}{32\sqrt{\pi}} \chi^4 e^{-2/3\chi} \sum_{k=0}^{\infty} (-\chi)^k \sum_{n=0}^{2k} \frac{\Gamma(9/2 + 2k - n) \Gamma(k + n + 1/2)}{n! (2k - n)! \Gamma(9/2)}. \quad (64)$$

To take into account the  $\omega$  dependence of the decay rate, one has to retain the higher-order terms in the expansion of  $\sin^2 x$  in the exponent in eq. (49). This gives, to leading order,

$$W(\chi) = W_0(|\varepsilon_0|) \frac{105}{32} \left(1 + \frac{5}{9\xi^2}\right) \chi^4 \exp \left[ -\frac{2}{3\chi} \left(1 - \frac{1}{15\xi^2}\right) \right]. \quad (65)$$

In the multi-photon regime ( $\omega t_2 = \xi^{-1} \gg 1$ ), one can find the probability of the process by calculating the integral in (49) in the saddle point approximation. The simpler expression (52) cannot be used in this case since a region of complex  $x$  with large modulus is important in the integral. The saddle point is found from the transcendental equation

$$\sinh^2 z - (\cosh z - z^{-1} \sinh z)^2 = \xi^{-2}, \quad (66)$$

where  $z = ix$ . Solving it approximately in the limit  $\xi \ll 1$ , we find

$$W \simeq W_0(|\varepsilon_0|) \frac{105}{32} \left( \frac{\hbar\omega}{2|\varepsilon_0|} \right)^4 \left( \ln \frac{2m|\varepsilon_0|\omega}{eE} \right)^{-9/2} \left( \frac{eE}{2m|\varepsilon_0|\omega} \right)^{\frac{2|\varepsilon_0|}{\hbar\omega} - 1/2}. \quad (67)$$

Thus, we see that, in accordance with the qualitative analysis of sec. 2, in the weak-field limit  $\chi \ll 1$  the probability of the field-induced  $\beta$  decay is exponentially suppressed in the low-frequency case and exhibits a power-law suppression in the high-frequency limit.

### 3.4 Neutrino mass effects

Up to now, in our calculations we have been neglecting the neutrino mass  $m_\nu$ . In the absence of external fields, the only characteristic of  $\beta$ -decay with the dimension of energy is the energy release  $\varepsilon_0$ , and it is always much greater than  $m_\nu c^2$ . This justifies the neglect of  $m_\nu$  in calculations of the total  $\beta$ -decay rates in the absence of external fields.

In the case of  $\beta$ -decay in a field of monochromatic wave, one can construct other quantities with the dimension of energy, which depend on the field strength and frequency and which have to be compared to  $m_\nu c^2$ . If any characteristics of the process should turn out to depend on the ratios of  $m_\nu c^2$  and these field-dependent parameters, study of  $\beta$ -decay in electromagnetic fields could provide important information on the value of the neutrino mass.<sup>3</sup> Using  $E_\nu = (p^2 + m_\nu^2)^{1/2}$  in eq. (47) and following the same steps as those that led to (49), we obtain [2]

$$W = -\frac{\sqrt{\pi i}}{2^4 \pi^4 \hbar^7 c^3} m^{3/2} |M_{fi}|^2 (m_\nu c^2)^2 (\hbar\omega)^{3/2} \times \int_{-\infty}^{\infty} \frac{dx}{(x + i0)^{5/2}} K_2 \left( -2i \frac{m_\nu c^2}{\hbar\omega} x \right) \exp \left\{ -i \left[ \delta \cdot x + \gamma \left( \frac{\sin^2 x}{x} - x \right) \right] \right\}. \quad (68)$$

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<sup>3</sup>Note that the effective neutrino mass  $m_\nu$  that can be probed in  $\beta$ -decay is expressed through the masses of neutrino mass eigenstates  $m_i$  and the elements of the leptonic mixing matrix  $U$  as  $m_\nu = \sum_i |U_{ei}|^2 m_i$ .

This equation can be studied by the same methods as those that were applied to eqs. (49) and (52). The analysis [3] shows that the dependence of the  $\beta$ -decay rate  $W$  on the neutrino mass has the same nature as its dependence on the field frequency  $\omega$ , i.e. it enters through the parameter  $(m_\nu c^2 t_x / \hbar)^2$ , where  $t_x$  is the characteristic time scale of the process. As was discussed above,  $t_x$  is different in different regimes. The parameter  $(m_\nu c^2 t_x / \hbar)^2$  turns out to be very small in all cases except for  $\varepsilon_0 < 0$ ,  $\chi \ll 1$ , when  $t_x$  is the tunneling time  $t_2$  defined in (35). In this case the corrections to the probabilities of field-induced  $\beta$ -decay due to non-vanishing neutrino mass can be sizable. However, the decay probabilities themselves are extremely small then, making the neutrino mass effects unobservable. Thus, unfortunately  $\beta$ -decay in external electromagnetic fields cannot tell us much about the neutrino mass, at least as far as the total probabilities are concerned. Note that interesting effects on the spectra of  $\beta$ -electrons may still be possible.

## 4 Forbidden $\beta$ decay in strong fields – lifting the forbiddenness

As was pointed out in sec. 2.2, one possible mechanism of modification of the total probabilities of quantum processes in electromagnetic fields is modification of their transition matrix elements. We shall discuss now a particular example of this – acceleration of forbidden nuclear  $\beta$ -decay due to the interaction of the parent or daughter nuclei with the field.

When considering allowed  $\beta$  decay in external fields we were taking into account only the interaction of the produced electron or positron with the field; the interaction of the involved nuclei was ignored because they are very heavy. In the case of forbidden  $\beta$ -decay, however, there exists an enhancement mechanism which requires taking the interaction of nuclei with the field into account. If the parent or daughter nucleus absorbs from the field (or emits into it) one or more photons in the course of  $\beta$ -decay, the angular momentum of the initial or final nuclear state may change, and the forbiddenness may be lifted.

Let us discuss this mechanism in more detail. Let a nucleus undergo a forbidden  $\beta$ -decay from its ground state  $|i\rangle$  to the ground state  $|f\rangle$  of the daughter nucleus. Assume that the parent nucleus has an excited state  $|1\rangle$  whose quantum numbers permit an allowed  $\beta$ -transition  $|1\rangle \rightarrow |f\rangle$ . The parent nucleus can then undergo a virtual transition to the state  $|1\rangle$  by absorbing one or more photons from the external field (or by emitting them into the field), followed by the allowed  $\beta$ -transition. An analogous situation will obtain if the daughter nucleus has an excited state  $|2\rangle$  with quantum numbers permitting an allowed  $\beta$ -decay  $|i\rangle \rightarrow |2\rangle$ , followed by the electromagnetic  $|2\rangle \rightarrow |f\rangle$  transition under the influence of the external field.<sup>4</sup>

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<sup>4</sup>We are assuming here that the excitation energy of the state  $|2\rangle$  is larger than the energy release  $\varepsilon_0$  of the ground state  $\rightarrow$  ground state  $\beta$ -transition. Otherwise the allowed  $\beta$ -decay  $|i\rangle \rightarrow |2\rangle$  followed by a cascade of  $\gamma$ -transitions  $|2\rangle \rightarrow |f\rangle$  would be possible even in the absence of the external field.

Let us now concentrate on the simple case of unique first-forbidden  $\beta$ -transitions for which the selection rules are  $\Delta J^{\Delta\pi} = 2^-$ . Higher-order forbidden transitions can be examined similarly.

For unique first-forbidden transitions, the state  $|1\rangle$  must be related to the ground state  $|i\rangle$  of the parent nucleus by the electric dipole ( $E1$ ) transition. The  $|1\rangle \rightarrow |f\rangle$  transition will then be the allowed Gamow-Teller transition with  $\Delta J^{\Delta\pi} = 1^+$ . An analogous situation would arise for transitions through the virtual state  $|2\rangle$  of the daughter nucleus. It is readily seen that any other possibility (e.g., electromagnetic  $M2$  transition and allowed Fermi transition with  $\Delta J^{\Delta\pi} = 0^+$ ) would result in a much smaller matrix element.

Let us estimate the matrix element  $M_{ind}$  of the induced process corresponding to the relaxation of the forbiddenness in external fields. We shall denote the energy differences between the states  $|1\rangle$ ,  $|i\rangle$  and  $|2\rangle$ ,  $|f\rangle$  as

$$\Delta\varepsilon_1 \equiv \varepsilon_1 - \varepsilon_i, \quad \Delta\varepsilon_2 \equiv \varepsilon_2 - \varepsilon_f. \quad (69)$$

The admixture of the state  $|1\rangle$  to the ground state  $|i\rangle$  in the external field of frequency  $\omega \ll \Delta\varepsilon_1$  is characterized by the parameter  $eE_0d_{1i}/\Delta\varepsilon_1$ , where  $d_{1i}$  is the dipole matrix element of the corresponding electromagnetic transition. Similarly, the admixture of the state  $|2\rangle$  to the ground state  $|f\rangle$  of the daughter nucleus is characterized by the parameter  $eE_0d_{2f}/\Delta\varepsilon_2$ . To estimate the matrix element  $M_{ind}$ , we replace the dipole matrix elements  $d_{1i}$  and  $d_{2f}$  by the nuclear radius  $R$  and obtain

$$M_{ind} \sim \frac{eE_0R}{\Delta\varepsilon_{1,2}} \cdot 1, \quad (70)$$

where for the estimate we have assumed that the matrix elements of allowed  $\beta$ -transitions are of order unity (in reality, for intermediate and heavy nuclei they are somewhat smaller).

Note that the “forbiddenness lifting parameter”  $eE_0R/\Delta\varepsilon_{1,2}$  does not depend on the field frequency  $\omega$  in the limit  $\omega \ll \Delta\varepsilon_{1,2}$ . This is in contrast with the results of refs. [12, 13], where it was claimed that the forbiddenness can be relaxed if the parameter  $z^{1/2} \equiv eE_0R/\omega$  becomes sizeable. As was discussed in sec. 2, the dependence of the total probabilities on the parameters that diverge in the limit  $\omega \rightarrow 0$  is not admissible from the physical point of view; it was shown in [3] that the results of refs. [12, 13] were a consequence of some unjustified approximations adopted in those papers, which, in particular, led to a breakdown of gauge invariance. As a result, the probabilities of forbidden nuclear  $\beta$ -decay in external electromagnetic fields were overestimated in [12, 13] by many orders of magnitude.

The dependence of the induced matrix element  $M_{ind}$  on the external field frequency can become important for large enough  $\omega$ , when  $\omega \sim \Delta\varepsilon_{1,2}$ ; in this case in eq. (70)  $\Delta\varepsilon_{1,2}$  in the denominator has to be replaced by  $\Delta\varepsilon_{1,2} \pm \omega$ . In particular, for  $\omega = \Delta\varepsilon_{1,2}$  a resonant enhancement of the process becomes possible; the denominator of  $M_{ind}$  in eq. (70) would then just contain the width of the corresponding excited state,  $\Gamma_1$  or  $\Gamma_2$ . Unfortunately, such a resonant enhancement would require  $\gamma$ -lasers, which do not exist at present.

Let us estimate the enhancement factor due to the relaxation of the forbiddenness of the  $\beta$ -transition in low-frequency fields. In the absence of the field, the matrix element of unique first-forbidden  $\beta$ -decay is  $M_0 \sim k_0 R/\hbar$ , where, as usual,  $k_0 = \sqrt{2m\varepsilon_0}$ . From eq. (70) we then obtain

$$\frac{M_{ind}}{M_0} \simeq \frac{eE_0\hbar}{\sqrt{2m\varepsilon_0}\Delta\varepsilon_{1,2}} \equiv \chi_*, \quad (71)$$

i.e. the parameter that governs the enhancement of the probability due to the lifting of the forbiddenness is  $\chi_* = \chi(2\varepsilon_0/\Delta\varepsilon_{1,2})$ . For the rate of the process in the case of relatively weak fields we then expect

$$W \simeq W_0(\varepsilon_0)[1 + a\chi_*^2 + b\chi_*^4 + \dots], \quad (72)$$

where  $a$  and  $b$  are numerical constants.

Let us now sketch the calculation of the decay rate (the details can be found in ref. [3]). Although all physical observables are gauge invariant, a judicious choice of the gauge can simplify the calculations greatly. For calculating the probability of forbidden  $\beta$ -decay in external electromagnetic fields, the scalar gauge (3) turns out that to be most convenient. The reason for this is that the corrections to the nuclear wave functions can then be described in the lowest order in the small parameter  $eE_0R/\Delta\varepsilon_{1,2}$  (see (70)). At the same time, the corresponding interaction parameter in the Coulomb gauge,<sup>5</sup>

$$\frac{e}{c} \frac{A_0}{M} \frac{p_{ni}}{(\varepsilon_n - \varepsilon_i)} = \frac{eE_0 d_{ni}}{\hbar\omega}, \quad (73)$$

is not in general small in low-frequency fields, and therefore it cannot be used as an expansion parameter in perturbation-theory calculations. (Here  $A_0 = cE_0/\omega$  is the amplitude of the vector-potential and  $n$  corresponds to an excited nuclear state). Moreover, unlike the scalar-gauge parameter  $eE_0R/\Delta\varepsilon_{1,2}$ , it is not suppressed for large values of  $\varepsilon_n - \varepsilon_i$ ; this means that in calculating nuclear wave functions in the presence of external fields one cannot constrain oneself to the contributions of only lowest-lying excited states and has to sum over the whole spectrum of nuclear excitations. Therefore, we choose to calculate in the scalar gauge (3).

The wave function of a non-relativistic electron in the scalar gauge takes the form [10]

$$\Psi'_{\mathbf{k}}(\mathbf{r}, t) = \exp \left\{ \frac{i}{\hbar} [\mathbf{k} - (e/c)\mathbf{A}(t)]\mathbf{r} - \frac{i}{2m\hbar} \int^t [\mathbf{k} - (e/c)\mathbf{A}(t')]^2 dt' \right\}, \quad (74)$$

where the prime refers to the quantities in the scalar gauge. Note that here  $\mathbf{A}(t)$  does not have the meaning of a vector-potential, which is zero in the scalar gauge (3); by definition, in eq. (74)  $\mathbf{A}(t) \equiv -c \int^t dt' \mathbf{E}(t')$ .

Comparing the electron wave functions (43) and (74), we find that they are related by

$$\Psi'_k(\mathbf{r}, t) = e^{-\frac{ie}{\hbar c} \mathbf{A}(t)\mathbf{r}} \Psi_k(\mathbf{r}, t). \quad (75)$$

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<sup>5</sup>Here we consider the parameter describing the modification of the wave function of the parent nucleus. The daughter nucleus can be considered similarly.



This can be readily understood by noting that the gauge transformation from the Coulomb gauge (2) to the scalar gauge (3) is  $A^\mu(t, \mathbf{x}) \rightarrow A^\mu(t, \mathbf{x}) + \partial^\mu \eta(t, \mathbf{x})$  with the gauge function  $\eta(t, \mathbf{x}) = \mathbf{A}(t) \cdot \mathbf{r}$ . The wave functions of particles of charge  $e$  should then transform as

$$\psi(t, \mathbf{r}) \rightarrow e^{-(ie/\hbar c)\eta(t, \mathbf{r})} \psi(t, \mathbf{r}). \quad (76)$$

The electron wave functions in the Coulomb gauge and scalar gauge thus have different coordinate dependence. For allowed  $\beta$ -transitions the electron wave function inside the nucleus is replaced by a constant, and therefore it does not matter whether the Coulomb-gauge or scalar-gauge expression is used. For calculating the rate of forbidden  $\beta$ -decays the wave function (74) should be employed.

The calculations [3] confirms the expected result (72) for the rate of unique first-forbidden  $\beta$  decay in the field of a strong electromagnetic wave. Since nuclear excitation energies  $\Delta\varepsilon_{1,2}$  are typically of the same order of magnitude as the energy release  $\varepsilon_0$ , and in all known cases are not smaller than  $\sim 1$  keV, for currently attainable fields the forbiddenness lifting term  $b\chi_*^2$  in (72) is extremely small, as is the phase space enhancement term  $a\chi^2$ . We discuss this point in more detail in the next section. Notice, however, that the numerical value of the coefficient  $a$  in (72) is significantly larger than in the case of allowed  $\beta$ -transitions: for unique first-forbidden  $\beta$ -decays the calculation [3] yields  $a = 315/8$ , to be compared with  $35/8$  in eq. (56).

## 5 Summary and discussion

We have considered, both qualitatively and quantitatively, the influence of the field of a strong electromagnetic wave on the characteristics of quantum processes with participation of non-relativistic charged particles. Our qualitative analysis has shown that the parameter that determines the influence of the external field on the differential characteristics of the processes (such as energy spectra and angular distributions of the final-state particles) is  $\xi = eE_0/(\sqrt{2m\varepsilon_0}\omega)$ . The processes of modification of these differential characteristics in external fields are essentially of classical nature.

At the same time, the parameter that governs the modification of the total probabilities of the processes (such as decay rates and total cross sections) is  $\chi = eE_0\hbar/(\sqrt{2m\varepsilon_0}2\varepsilon_0)$ . It describes the energy  $\delta\varepsilon_D$  obtained by the charged particle from the field (or given to the field) over the distances of order of the particle's de Broglie wavelength. In relatively weak fields, when  $\chi \ll 1$ , this energy is given by  $\delta\varepsilon_D \simeq \chi\varepsilon_0$ , whereas in the strong field limit ( $\chi \gg 1$ ) one has  $\delta\varepsilon_D \simeq \chi^{2/3}\varepsilon_0$ . The fractional power of  $\chi$  enters in the latter case because of the field dependence of the charged particle's de Broglie wavelength. The process of energy exchange between the produced particle and the external field in the process of the formation of the particle (and therefore the modification of the total probability of the process) is inherently quantum in its nature.

We also estimated the total probabilities of quantum processes in external electromagnetic fields, considering nuclear  $\beta$ -decay as an example. Our estimates were made in the limits  $\chi \ll 1$  and  $\chi \gg 1$ . In the latter case the decay rate for a  $\beta$ -active nucleus ( $\varepsilon_0 > 0$ ) and the rate of the field-induced  $\beta$ -decay of the daughter nucleus which is stable in the absence of the external field ( $\varepsilon_0 < 0$ ) are practically the same. Examples of such transitions are  ${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_e$  ( $\varepsilon_0 > 0$ ) and  ${}^3\text{He} \rightarrow {}^3\text{H} + e^+ + \nu_e$  ( $\varepsilon_0 < 0$ ). In the weak-field limit the induced decay in the case  $\varepsilon_0 < 0$  is predicted to be exponentially suppressed in the low-frequency (tunneling) limit  $\omega t_2 \ll 1$ , where  $t_2$  is the tunneling time defined in (35), and power-suppressed in the multi-photon regime  $\omega t_2 \gg 1$ .

The dependence of the total probabilities of quantum processes on the field frequency  $\omega$  comes through the expansion in powers of the parameter  $(\omega t_x)^2$ , where  $t_x$  is the characteristic formation time of the produced charged particle. At the same time, as was mentioned above, the energy obtained by the particle in the process of its formation is given by the work of the field on the particle over the distance of order of the particle's formation length  $l_x$ :  $\delta\varepsilon \simeq eE_0 l_x$ . The quantities  $l_x$  and  $t_x$  are different in different regimes:

- Weak fields :

$$\begin{aligned} \varepsilon_0 > 0 : \quad l_x &\simeq \frac{\hbar}{2\sqrt{2m\varepsilon_0}}, \quad t_x \simeq \hbar/2\varepsilon_0 \quad \Rightarrow \quad \omega t_x = \hbar\omega/2\varepsilon_0 = 1/\delta, \\ \varepsilon_0 < 0 : \quad l_x &\simeq \frac{|\varepsilon_0|}{eE_0}, \quad t_x = \frac{\sqrt{2m|\varepsilon_0|}}{eE_0} \quad \Rightarrow \quad \omega t_x = \sqrt{2m|\varepsilon_0|}\omega/eE_0 = 1/\xi, \end{aligned}$$

- Strong fields :

$$l_x \simeq \left( \frac{\hbar^2}{8meE_0} \right)^{1/3}, \quad t_x \simeq \left( \frac{m\hbar}{e^2 E_0^2} \right)^{1/3} \quad \Rightarrow \quad \omega t_x = (\hbar\omega/2\varepsilon_0)\chi^{-2/3}.$$

We have also discussed a simple method of calculating the total probabilities of quantum processes with participation of non-relativistic charged particles in the field of an electromagnetic wave, considering nuclear  $\beta$ -decay as an example. The method does not rely on a summation of partial probabilities with absorption from the field or emission into it of all possible numbers of photons; instead, the total probabilities are calculated directly. The results of the direct calculations in the cases of allowed and forbidden  $\beta$ -decay fully confirmed the estimates made in sec. 2, often even including the values of numerical coefficients.

How large can actually the effects of strong electromagnetic fields on  $\beta$ -decay rates be? The parameter  $\chi$  that determines the modification of the rates can be written as

$$\chi = \frac{eE_0\hbar}{\sqrt{2m\varepsilon_0}2\varepsilon_0} = \frac{E_0}{E_c} \left( \frac{mc^2}{2\varepsilon_0} \right)^{3/2}, \quad (77)$$

where

$$E_c = \frac{m^2 c^3}{e\hbar} = 1.323 \times 10^{16} \text{ V/cm} \quad (78)$$

is the QED critical field strength (the so-called Schwinger field). The most powerful present-day lasers can reach the field intensities up to  $I \sim 10^{22}$  W/cm<sup>2</sup>, and in the near future probably the intensities  $I \sim 10^{24}$  W/cm<sup>2</sup> will become available. From the formula

$$E_0(\text{V/cm}) \simeq 20 \sqrt{I(\text{W/cm}^2)} \quad (79)$$

we then find that this corresponds to the field strengths  $E_0/E_c \sim 10^{-4} - 10^{-3}$ . For tritium  $\beta$ -decay from eq. (77) we then obtain

$$\chi \simeq 5 \times 10^{-3} - 5 \times 10^{-2}. \quad (80)$$

Taking into account that for small  $\chi$  the correction to the decay rate is of order  $\chi^2$ , we see that the field effect on tritium  $\beta$ -decay is too small to be observable.

Since  $\chi^2$  scales as  $\varepsilon_0^{-3}$ , one can expect significantly stronger effects for  $\beta$ -decays with smaller energy release. In particular, for unique first forbidden  $\beta$ -decay  $^{187}\text{Re}(\frac{5}{2}^+) \rightarrow ^{187}\text{Os}(\frac{1}{2}^-)$  one has  $\varepsilon_0 \simeq 2.64$  keV, and the parameter  $\chi$  can formally take values  $\chi \sim 1$  even for present-day lasers. Does that mean that we can already observe strong effects of the laser fields on nuclear  $\beta$ -decay?

Unfortunately, in reality the situation is not that promising at the moment. First, for nuclei with small energy release (and especially for forbidden  $\beta$ -decays) the decay rates are extremely small, and even sizeable corrections to them are not easily observable: the lifetime of  $^{187}\text{Re}$ , for example, is  $\sim 5 \times 10^{10}$  yr. The situation is complicated by the fact that the powerful lasers have very low pulse duration (in the femtosecond range) and repetition frequency which is typically  $\sim 10^{-3}$  Hz. Finally, except for  $\omega \gg \omega_{at}$  where  $\omega_{at}$  are characteristic atomic frequencies, the atomic electrons would screen the external fields, greatly reducing their strengths at the nucleus, and to observe the field effect one would first have to ionize the atom of the  $\beta$ -active element.

In contrast to this, laser field effects on energy spectra and angular distributions of electrons and positrons emitted in nuclear  $\beta$ -decay can be quite significant. In addition, strong electromagnetic fields can still influence sizeably the total probabilities of atomic and molecular processes with emission of low-energy electrons. An example of such a process is a photo-ionization in two fields, one of which is a weak field with the energy of the quantum  $\hbar\Omega$  slightly above (or slightly below) the ionization potential  $I$  and the other – an intense field with the frequency satisfying  $\hbar\omega \ll I$ .

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## Appendix A: Validity of approximations

We consider here the validity conditions for the approximations adopted in our calculations of the total probabilities of quantum processes.

The dipole approximation implies that the wavelength of the external electromagnetic field is assumed to be large compared to the characteristic length parameters  $l_x$  involved in the problem:  $\lambda = 2\pi c/\omega \gg l_x$ . Let us consider this condition in various regimes.

(a)  $\chi \ll 1$ ,  $\varepsilon_0 > 0$ :  $l_x \simeq \lambda_D/2 = \hbar/(2\sqrt{2m\varepsilon_0})$ . The dipole approximation requires

$$\frac{\hbar\omega}{2\varepsilon_0} \ll 4\pi \left( \frac{mc^2}{2\varepsilon_0} \right)^{1/2} \gg 1, \quad (\text{A1})$$

which is certainly satisfied for  $\hbar\omega \lesssim \varepsilon_0$ .

(b)  $\chi \ll 1$ ,  $\varepsilon_0 < 0$ :  $l_x = |\varepsilon_0|/eE_0$ . The dipole approximation is valid provided that

$$\frac{\hbar\omega}{2|\varepsilon_0|} \ll 4\pi\chi \left( \frac{mc^2}{2|\varepsilon_0|} \right)^{1/2}. \quad (\text{A2})$$

(c)  $\chi \gg 1$ ,  $\varepsilon_0 > 0$  or  $< 0$ :  $l_x = [\hbar^2/(8meE_0)]^{1/3}$ . The validity condition is

$$\frac{\hbar\omega}{2|\varepsilon_0|} \ll 4\pi\chi^{1/3} \left( \frac{mc^2}{2|\varepsilon_0|} \right)^{1/2}, \quad (\text{A3})$$

which is satisfied with a large margin.

Next, we discuss the non-relativistic approximation. In the weak-field limit ( $\chi \ll 1$ ) the standard condition is

$$\varepsilon_0 \ll mc^2. \quad (\text{A4})$$

However, in the strong field limit  $\chi \gg 1$  one also has to make sure that the energy obtained from the field by the charged particle during its formation does not take it out of the non-relativistic domain:  $eE_0 l_x \ll mc^2$ . Taking into account that in the strong-field regime  $l_x = [\hbar^2/(8meE_0)]^{1/3}$ , we arrive at the condition

$$E_0 \ll E_c = 1.323 \times 10^{16} \text{ V/cm}. \quad (\text{A5})$$

In our calculations of the rates of nuclear  $\beta$ -decay in external fields we were neglecting the final-state interaction of the produced electron or positron with the Coulomb field of the nucleus. This is permissible when the Coulomb energy  $Ze^2/l_x = \alpha Z\hbar c/l_x$  is small compared to the characteristic energy  $\varepsilon_x$  of the process. Consider now the Coulomb parameter  $\alpha Z\hbar c/l_x \varepsilon_x$  in various regimes.

(a)  $\chi \ll 1$ ,  $\varepsilon_0 > 0$ :  $l_x \simeq \hbar/(2\sqrt{2m\varepsilon_0})$ ,  $\varepsilon_x \simeq \varepsilon_0$ . In this case

$$\alpha Z\hbar c/\varepsilon_x l_x \simeq 4\alpha Zc/v_0 = 4\alpha Zc(m/2\varepsilon_0)^{1/2}, \quad (\text{A6})$$

which is (up to the factor 4) just the standard Coulomb parameter. As an example, for  $\beta$ -decay of  ${}^3\text{H}$  we have  $Z = 2$ ,  $(m/2\varepsilon_0)^{1/2} \simeq 3.7$ ,  $\alpha Z\hbar c/\varepsilon_x l_x \simeq 0.2$ .

$$(b) \quad \chi \ll 1, \varepsilon_0 < 0: \quad l_x = |\varepsilon_0|/eE, \quad \varepsilon_x = |\varepsilon_0|,$$

$$\alpha Z\hbar c/\varepsilon_x l_x \simeq 4\alpha Zc(m/2|\varepsilon_0|)^{1/2} \chi. \quad (\text{A7})$$

$$(c) \quad \chi \gg 1, \varepsilon_0 > 0 \quad \text{or} \quad < 0: \quad l_x = [\hbar^2/(8meE_0)]^{1/3}, \quad \varepsilon_x = \chi^{2/3}|\varepsilon_0|,$$

$$\alpha Z\hbar c/\varepsilon_x l_x \simeq 4\alpha Z(m/2|\varepsilon_0|)^{1/2} \chi^{-1/3}. \quad (8)$$

## Appendix B: Some useful formulas

Here we collect some formulas which have been used in calculation done in secs. 3 and 4.

An integral representation for  $J_0(z)$ :

$$J_0(z) = \frac{1}{\pi} \int_0^\pi e^{iz \cos \theta} d\theta. \quad (\text{B1})$$

Gradshteyn & Ryzhik [11], 6.631(6):

$$\int_0^\infty x^{\nu+1} e^{\pm i\alpha x^2} J_\nu(\beta x) dx = \frac{\beta^\nu}{2\alpha^{\nu+1}} \exp \left[ \pm i \left( \frac{\nu+1}{2} \pi - \frac{\beta^2}{4\alpha} \right) \right]. \quad (\text{B2})$$

$$[\alpha > 0, \quad -1 < \text{Re}\nu < 1/2, \quad \beta > 0].$$

Gradshteyn & Ryzhik [11], 3.382.7:

$$\int_{-\infty}^\infty \frac{e^{-ipx}}{(\beta - ix)^\nu} dx = \begin{cases} \frac{2\pi p^{\nu-1} e^{-\beta p}}{\Gamma(\nu)}, & p > 0 \\ 0, & p < 0 \end{cases}. \quad (\text{B3})$$

$$[\text{Re}\nu > 0, \quad \text{Re}\beta > 0].$$

For an electromagnetic wave with the electric field strength  $\mathbf{E}(t) = \{E_0 \sin \omega t, -E_0 \cos \omega t, 0\}$  the electron wave function in the Coulomb gauge (43) is

$$\Psi_{\mathbf{k}}(\mathbf{r}, t) = \exp \left\{ \frac{i}{\hbar} \mathbf{k} \mathbf{r} - \frac{i}{\hbar} \left[ \left( \frac{k^2}{2m} + \frac{e^2 A_0^2}{2mc^2} \right) t - \frac{eA_0}{mc\omega} (k_x \sin \omega t - k_y \cos \omega t) \right] \right\}, \quad (\text{B4})$$

where  $A_0 = E_0 c/\omega$ .

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